SUPERCONVERGENCE OF IMMERSED FINITE VOLUME METHODS FOR INTERFACE PROBLEMS

WAIXIANG CAO†, XU ZHANG‡, ZHIMIN ZHANG§, AND QINGSONG ZOU¶

Abstract. In this paper, we introduce a class of high order immersed finite volume methods (IFVM) for one dimensional interface problems. We show that the IFVM converge optimally in $H^1$- and $L^2$- norms. We also prove that the IFVM inherit all the desired superconvergence results from the standard finite volume methods. All theoretical results are confirmed by numerical experiments.

Key words. superconvergence, immersed finite volume method, interface problems, generalized orthogonal polynomials

1. Introduction. Interface problems arise in many simulations in science and engineering that involve multi-physics and multi-materials. Classical numerical methods, such as finite element methods (FEM) [8, 17, 38], and finite volume methods (FVM) [6, 9, 21, 34, 35] usually require solution meshes to fit the interface; otherwise, the convergence may be impaired. The immersed finite element methods (IFEM) [1, 2, 27] are a class of FEM that relax the body-fitting requirement, hence Cartesian meshes can be used for solving interface problems with arbitrary interface geometry. The key ingredient of IFEM is to design some special basis functions on interface elements that can capture the non-smoothness of the exact solution. Recently, this immersed idea has also been used in a variety of numerical schemes such as conforming FEM [23, 28, 29], nonconforming FEM [26, 30, 31], discontinuous Galerkin methods [25, 32, 44], and FVM [20, 24].

The use of structured mesh, especially Cartesian meshes, often leads to some superconvergence phenomenon. The superconvergence is a phenomenon that the order of convergence at certain points surpass the maximum order of convergence of the numerical schemes. There has been a growing interest in the study of superconvergence, for example, finite element methods [4, 7, 16, 33, 36, 40], finite volume methods [9, 11, 15, 18, 42], discontinuous Galerkin and local discontinuous Galerkin methods [3, 12, 13, 14, 22, 39, 43].

In this article, we first introduce a class of high order IFVM for one dimensional interface problems. Thanks to the unified construction of FVM schemes in [11, 45] and the generalized orthogonal polynomials developed in [10], we can develop the high order IFVM in a systematical approach. To be more specific, we adopt the standard

†The work of W. Cao was supported in part by the NSFC grant 11501026, and the China Postdoctoral Science Foundation grant 2016T90027, 2015M570026. The work of Z. Zhang was supported in part by the NSFC grants 11471031, 91430216, and U1530401; and NSF grant DMS-1419040. The work of Q. Zou was supported in part by the NSFC grants 11571384 and 11428103, by Guangdong NSF grant 2014A030313179 and by Fundamental Research Funds for the Central Universities grant 16lgjc80.

‡School of Data and Computer Science, Sun Yat-Sen University, Guangzhou, Guangdong 51006, China (caowx5@mail.sysu.edu.cn).

§Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA (xuzhang@math.msstate.edu).

¶Beijing Computational Science Research Center, Beijing 100193, China (zmzhang@csrc.ac.cn); and Department of Mathematics, Wayne State University, Detroit, MI 48202, USA (zhang@math.wayne.edu).

¶School of Data and Computer Science, Sun Yat-Sen University, Guangzhou, Guangdong 51006, China (mcszqs@mail.sysu.edu.cn).
$p$-th degree IFE spaces [1, 2, 10] as our trial function space. Using the roots of generalized Legendre polynomials, known as generalized Gauss points, as the control volume, we construct the test function space as the piecewise constant corresponding to the dual meshes. The advantage of our IFVM is that it does not require the mesh to be aligned with the interface, and it inherits all the desired properties of the classical FVM such as local conservation of flux.

The main focus of this article is the error analysis of IFVM, especially the superconvergence analysis. By establishing the inf-sup condition and continuity of the bilinear form, we prove that our IFVM converge optimally in $H^1$-norm. As for the superconvergence, we prove that the immersed finite volume (IFV) solution is superconvergent of the order $O(h^{p+2})$ at the generalized Lobatto points on both non-interface and interface elements, and the flux error is superconvergent at the generalized Gauss points of the order $O(h^{p+1})$. The error of IFV solution and the Gauss-Lobatto projection is superclose. In particular, for the diffusion interface problem, we show that the convergence rate of both the solution error at nodes and the flux error at Gauss points can be enhanced to $O(h^{2p})$. All these results are consistent with the superconvergence analysis of the standard FVM in [11].

However, there is a significant difference in the superconvergence analysis of IFVM compared with the analysis of standard FVM [11]. Due to the low global regularity of the exact solution, the standard approach using the Green function cannot be directly applied to the IFV for interface problems. The key ingredient in the analysis is the construction of generalized Lobatto points and a specially designed interpolation function. That is, we first choose a class of generalized Lobatto polynomials as our basis functions that satisfy both orthogonality and interface jump conditions, then we use these orthogonal basis function to design a special interpolant of the exact solution which is superclose to the IFV solution. The supercloseness of the interpolation and the IFV solution yields the desired superconvergence results for the IFV solution.

The rest of the paper is organized as follows. In Section 2 we recall the generalized orthogonal polynomials and present the high order IFVM for interface problems in one-dimensional setting. In Section 3 we provide a unified analysis for the inf-sup condition and establish the optimal convergence in $H^1$ norm. In Section 4, we study the superconvergence property of IFVM. We identify and analyze superconvergence points for the IFV solution at both interface and non-interface elements. Finally, some numerical examples are presented in Section 5.

In the rest of this paper, we use the notation “$A \lesssim B$” to denote $A$ can be bounded by $B$ multiplied by a constant independent of the mesh size. Moreover, “$A \sim B$” means “$A \lesssim B$” and “$B \lesssim A$”.

2. Interface Problems and Immersed Finite Volume Methods. Assume that $\Omega = (a, b)$ is an open interval in $\mathbb{R}$. Let $\alpha \in \Omega$ be an interface point such that $\Omega^- = (a, \alpha)$ and $\Omega^+ = (\alpha, b)$. Consider the following one-dimensional elliptic interface problem

\begin{align}
- (\beta u')' + \gamma u' + cu &= f, \quad x \in \Omega^- \cup \Omega^+, \\
\end{align}

\begin{align}
u(a) = u(b) &= 0.
\end{align}

Here, the coefficients $\gamma$ and $c$ are assumed to be constants. The diffusion coefficient $\beta$ has a finite jump across the interface. Without loss of generality, we assume it is a
piecewise constant function

\[
\beta(x) = \begin{cases} 
\beta^{-}, & \text{if } x \in \Omega^{-}, \\
\beta^{+}, & \text{if } x \in \Omega^{+},
\end{cases}
\]

where \(\beta_0 = \min\{\beta^{+}, \beta^{-}\} > 0\). At the interface \(\alpha\), the solution is assumed to satisfy the interface jump conditions

\[
u(\alpha) = 0, \quad \beta u'(\alpha) = 0,
\]

where \([v(\alpha)] = \lim_{x \to \alpha^+} v(x) - \lim_{x \to \alpha^-} v(x)\).

### 2.1. Generalized orthogonal polynomials.

We briefly review the generalized Legendre and Lobatto polynomials developed in [10]. These generalized orthogonal polynomials will be used to form the trial function space in the IFVM.

Let \(\tau = [-1, 1]\) be the reference interval, and \(P_n(\xi)\) be the standard Legendre polynomial of degree \(n\) on \(\tau\) satisfying the following orthogonality condition

\[
\int_{-1}^{1} P_m(\xi) P_n(\xi) d\xi = \frac{2}{2n+1} \delta_{mn}.
\]

Define a family of Lobatto polynomials \(\{\psi_n\}\) on \(\tau\) as follows

\[
\psi_0(\xi) = \frac{1 - \xi}{2}, \quad \psi_1(\xi) = \frac{1 + \xi}{2}, \quad \psi_n(\xi) = \int_{-1}^{\xi} P_{n-1}(t) dt, \quad n \geq 2.
\]

The generalized Legendre polynomials \(\{L_n\}\) on \(\tau\) with a discontinuous weight is defined as

\[
(L_n, L_m)_w := \int_{-1}^{1} w(\xi) L_n(\xi) L_m(\xi) d\xi = c_n \delta_{mn}.
\]

where \(w(\xi) = \frac{1}{\beta(\xi)}\) and

\[
\beta(\xi) = \begin{cases} 
\beta^{-}, & \text{if } \xi \in \tau^{-} = (-1, \hat{\alpha}), \\
\beta^{+}, & \text{if } \xi \in \tau^{+} = (\hat{\alpha}, 1),
\end{cases}
\]

The generalized Lobatto polynomials \(\{\phi_n\}\) can be constructed in a similar manner as (2.6) as follows:

\[
\phi_0(\xi) = \begin{cases} 
(1 - \hat{\alpha})\beta^{-} + (\hat{\alpha} - \xi)\beta^{+}, & \text{in } \tau^{-}, \\
(1 - \xi)\beta^{-} + (1 + \hat{\alpha})\beta^{+}, & \text{in } \tau^{+}.
\end{cases}
\]

\[
\phi_1(\xi) = \begin{cases} 
(1 + \xi)\beta^{-} + (1 - \hat{\alpha})\beta^{+}, & \text{in } \tau^{-}, \\
(\xi - \hat{\alpha})\beta^{-} + (1 + \hat{\alpha})\beta^{+} & \text{in } \tau^{+}.
\end{cases}
\]

\[
\phi_n(\xi) = \int_{-1}^{\xi} w(t) L_{n-1}(t) dt, \quad n \geq 2,
\]

These generalized orthogonal polynomials can be used as local basis functions on interface element, as they satisfy both the orthogonality and interface jump conditions:

\([\phi_n(\hat{\alpha})] = 0, \quad [\beta \phi_n^{(j)}(\hat{\alpha})] = 0, \quad \forall j = 1, 2, \cdots, n.\)
Note that the generalized Legendre polynomials are polynomials, but the generalized Lobatto polynomials are piecewise polynomials. As pointed out in [10], the generalized orthogonal polynomials can be explicitly constructed. In Figure 2.1, we plot the first few generalized orthogonal polynomials for $\hat{\beta} = [1, 5]$, and the reference interface point $\hat{\alpha} = 0.15$.

![Generalized Lobatto (left) and Legendre (right) polynomials with interface $\hat{\alpha} = 0.15$](image)

**Fig. 2.1.** Generalized Lobatto (left) and Legendre (right) polynomials with interface $\hat{\alpha} = 0.15$

### 2.2. Immersed finite volume methods.

In the subsection, we introduce the immersed finite volume methods for solving the interface problem (2.1) - (2.4). Consider the following partition of $\Omega$, independent of interface

\begin{equation}
(2.12) \quad a = x_0 < x_1 < \cdots < x_{k-1} < \alpha < x_k < \cdots < x_N = b.
\end{equation}

For a positive integer $N$, let $Z_N := \{1, \cdots, N\}$ and for all $i \in Z_N$, we denote $\tau_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$, $h = \max_i h_i$. Let $T = \{\tau_i\}_{i=1}^N$ be a partition of $\Omega$, and we suppose the partition is shape regular, i.e., there exists a constant $C$ such that

\[ h \leq C h_i, \quad \forall i \in Z_N. \]

We call the element $\tau_k$ the interface element since it contains the interface point $\alpha$, and the rest of elements $\tau_i$, $i \neq k$ noninterface elements.

The basis functions of the trial function space is constructed using the (generalized) Lobatto polynomials. In fact, we define the basis functions in each element $\tau_i$, $i \in Z_N$ as

\begin{equation}
(2.13) \quad \phi_{i,n}(x) = \begin{cases} 
\psi_n(\xi) = \psi_n \left( \frac{2x-x_{i-1}-x_i}{h_i} \right), & i \neq k, \\
\phi_n(\xi) = \phi_n \left( \frac{2x-x_{k-1}-x_k}{h_k} \right), & i = k.
\end{cases}
\end{equation}

Then corresponding trial function space is defined by

\begin{equation}
(2.14) \quad U_T := \{ v \in C(\Omega) : v|_{\tau_i} \in \text{span}\{\phi_{i,n} : n = 0, 1, \cdots, p\}, v(a) = v(b) = 0\}.
\end{equation}

Obviously, $\dim U_T = Np - 1$.

Next we present the dual partition and its corresponding test function space. It has been shown in [10] that the generalized Legendre polynomials $\{L_n\}$ and generalized Lobatto polynomials $\{\phi_n\}$ have same numbers of roots as the standard Legendre
polynomials \( \{P_n\} \) and Lobatto polynomials \( \{\psi_n\} \). Let

\[
(2.15) \quad P_n(x) = \begin{cases} 
P_n(\xi) = P_n\left(\frac{2x-x_i-1}{h_i}\right), & i \neq k, \\
L_n(\xi) = L_n\left(\frac{2x-x_i-1}{h_k}\right), & i = k.
\end{cases}
\]

We denote by \( g_{i,j}, j \in \mathbb{Z}_n \) the (generalized) Gauss points of degree \( n \) in \( \tau_i \). That is, the \( n \) roots of \( P_n \).

With these Gauss points, we construct a dual partition

\[
\mathcal{T}' = \{\tau'_{1,0}, \tau'_{N,p}\} \cup \{\tau'_{i,j} : (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}\},
\]

where

\[
\tau'_{1,0} = [a, g_{1,1}], \tau'_{N,p} = [g_{N,p}, b], \tau'_{i,j} = [g_{i,j}, g_{i,j+1}],
\]

here

\[
p_i = \begin{cases} 
p & \text{if } i \in \mathbb{Z}_N - 1 \\
p - 1 & \text{if } i = N \end{cases} \quad \text{and} \quad g_{i,p+1} = g_{i+1,1}, \forall i \in \mathbb{Z}_N - 1.
\]

The test function space \( V_{\mathcal{T}'} \) consists of the piecewise constant functions with respect to the partition \( \mathcal{T}' \), which vanish on the intervals \( \tau'_{i,0} \cup \tau'_{N,p} \). In other words,

\[
V_{\mathcal{T}'} = \text{Span} \{\varphi_{i,j} : (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}\},
\]

where \( \varphi_{i,j} = \chi_{[g_{i,j}, g_{i,j+1}]} \) is the characteristic function on the interval \( \tau'_{i,j} \). We find that \( \dim V_{\mathcal{T}'} = Np - 1 = \dim U_{\mathcal{T}} \).

The IFVM for solving (2.1) - (2.4) is: find \( u_{\mathcal{T}} \in U_{\mathcal{T}} \) such that

\[
\beta(g_{i,j})u'_{\mathcal{T}}(g_{i,j}) - \beta(g_{i,j+1})u'_{\mathcal{T}}(g_{i,j+1}) + \int_{g_{i,j}}^{g_{i,j+1}} (\gamma u'_{\mathcal{T}}(x) + cu_{\mathcal{T}}(x))dx \\
= \int_{g_{i,j}}^{g_{i,j+1}} f(x)dx, \quad \forall (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}.
\]

Given a function \( v_{\mathcal{T}'} \in V_{\mathcal{T}'} \), \( v_{\mathcal{T}'} \) can be represented as

\[
v_{\mathcal{T}'} = \sum_{i=1}^{N} \sum_{j=1}^{p_i} v_{i,j} \varphi_{i,j},
\]

where \( v'_{i,j} \)s are constants. Multiplying (2.16) with \( v_{i,j} \) and then summing up all \( i, j \), we obtain

\[
\sum_{i=1}^{N} \sum_{j=1}^{p_i} v_{i,j} \left( (\beta u'_{\mathcal{T}})(g_{i,j}) - (\beta u'_{\mathcal{T}})(g_{i,j+1}) + \int_{g_{i,j}}^{g_{i,j+1}} (\gamma u'_{\mathcal{T}}(x) + cu_{\mathcal{T}}(x))dx \right) = \int_{a}^{b} f(x)v_{\mathcal{T}'}(x)dx,
\]

or equivalently,

\[
\sum_{i=1}^{N} \sum_{j=1}^{p_i} [v_{i,j}] (\beta u'_{\mathcal{T}})(g_{i,j}) + \sum_{i=1}^{N} \sum_{j=1}^{p_i} v_{i,j} \left( \int_{g_{i,j}}^{g_{i,j+1}} (\gamma u'_{\mathcal{T}}(x) + cu_{\mathcal{T}}(x))dx \right) = \int_{a}^{b} f(x)v_{\mathcal{T}'}(x)dx,
\]

where \( [v_{i,j}] = v_{i,j} - v_{i,j-1} \) is the jump of \( v \) at the point \( g_{i,j}, (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_p \) with \( v_{1,0} = 0, v_{N,p} = 0 \) and \( v_{i,0} = v_{i-1,p}, 2 \leq i \leq N \).
The bilinear form of IFVM can be written as
\begin{equation}
(2.17)
a(u, v_T) = \sum_{i=1}^{N} \sum_{j=1}^{p} \beta v_{i,j} (g_{i,j}) u'(g_{i,j}) + \sum_{i=1}^{N} \sum_{j=1}^{p} v_{i,j} \left( \int_{g_{i,j}}^{g_{i,j+1}} (\gamma u'(x) + c u(x))dx \right),
\end{equation}
for all $u \in H^1_0(\Omega), v_T \in V_T$. Then our IFVM for the interface problem (2.1) - (2.4) can be rewritten as: Find $u_T \in U_T$ such that
\begin{equation}
(2.18)
a(u_T, v_T) = (f, v_T), \quad \forall v_T \in V_T.
\end{equation}

3. Convergence analysis. In this section, we derive the error estimation for IFVM. Following the same idea as in [11], we first prove the inf-sup condition and continuity of the IFVM, and then use them to establish the optimal convergence rate of the IFV approximation.

3.1. Inf-sup condition. We begin with some preliminaries. First, for any sub-domain $\Lambda \subset \Omega$, where $\Lambda^{\pm} = \Lambda \cap \Omega^{\pm}$, we define the following Sobolev spaces for $m \geq 1$ and $q \geq 1$ in $\Lambda$ as
\begin{equation}
(3.1)
\tilde{W}^{m,q}_{\beta}(\Lambda) = \left\{ v \in C(\Lambda): v|_{\Lambda^{\pm}} \in W^{m,q}(\Lambda^{\pm}), v|_{\partial \Omega \cap \Lambda} = 0, \quad \begin{bmatrix} \beta v^{(j)}(\alpha) \end{bmatrix} = 0, \quad j = 1, 2, \ldots, m \right\}
\end{equation}
equipped the norm and semi-norm
\[ \|v\|_{m,q,\Lambda}^q = \|v\|_{m,q,\Lambda^{-}} + \|v\|_{m,q,\Lambda^{+}}, \quad |v|_{m,q,\Lambda}^q = |v|_{m,q,\Lambda^{-}} + |v|_{m,q,\Lambda^{+}}. \]
If $\Lambda = \Omega$, we usually write $\|\cdot\|_{m,q}$ instead of $\|\cdot\|_{m,q,\Omega}$, and $|\cdot|_{m}$ instead of $|\cdot|_{m,2}$ when $q = 2$ for simplicity. Second, we define a discrete energy norm for all $v \in H^1(\Omega)$ by
\[ \|v\|_G^2 = \|v\|_{1}^2, \quad |v|_G^2 = \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} (\beta v'(g_{i,j}))^2. \]
Here $A_{i,j}$ is the weights of the Gauss quadrature
\[ Q_p(F) = \sum_{j=1}^{p} A_{i,j} F(g_{i,j}) \]
for computing the integral
\[ I(F) = \int_{\tau_i} w(x) F(x)dx = \int_{\tau_i} \frac{1}{\beta(x)} F(x)dx. \]

For all $v_T \in V_h$, $v_T = \sum_{i=1}^{N} \sum_{j=1}^{p_i} v_{i,j} \varphi_{i,j}$, we let
\[ |v_T|_{1,T}^2 = \sum_{i=1}^{N} \sum_{j=1}^{p_i} h_i^{-1} |v_{i,j}|^2, \quad \|v_T\|_{0,T}^2 = \sum_{i=1}^{N} \sum_{j=1}^{p_i} h_i v_{i,j}^2, \]
and
\[ \|v_T\|_T^2 = |v_h|_{1,T}^2 + \|v_T\|_{0,T}^2. \]

Also, we define a linear mapping \( \Pi_h : U_T \to V_T \), by
\[ v_T' = \Pi_h v_T = \sum_{i=1}^{N} \sum_{j=1}^{p_i} v_{i,j} \psi_{i,j}, \]
where the coefficients \( v_{i,j} \) are determined by the constraints
\[ \begin{bmatrix} v_{i,j} \end{bmatrix} = A_{i,j} (\beta v_T') (g_{i,j}), \quad (i,j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}. \quad (3.2) \]

**Lemma 3.1.** For any \( v_T \in U_T \), there holds
\[ \|v_T\|_1 \sim \|v_T\|_G, \quad \|\Pi_h v_T\|_{T'} \lesssim \|v_T\|_1. \quad (3.3) \]

**Proof.** Noticing that \( (\beta v_T')^2 \in P_{2p-2} \) for all \( v_T \in U_T \), and the \( p \)-point Gauss quadrature is exact for all polynomials of degree up to \( 2p - 1 \), we obtain
\[ \sum_{i=1}^{N} \int_{\tau_i} \beta(x)(v_T')^2(x)dx = \sum_{i=1}^{N} \int_{\tau_i} w(x)(v_T')^2(x)dx = \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} (\beta v_T')^2 (g_{i,j}). \quad (3.4) \]

Then the first inequality (3.3) follows.

Denote \( v_{1,0} = 0 \). It follows from a direct calculation that
\[ v_{i,j} = \sum_{m=1}^{i} \sum_{n=0}^{j} [v_{m,n}], \]
and thus
\[ v_{i,j}^2 \leq p(b-a) \sum_{m=1}^{N} \sum_{n=0}^{p} h_{m}^{-1} [v_{m,n}]^2. \]

Then
\[ \|\Pi_h v_T\|_{0,T'} \leq p(b-a)\|\Pi_h v_T\|_{1,T}. \]

On the other hand, for all \( v_T \in U_T \), the derivative \( \beta v_T' \in P_{p-1}(\tau_i), i \in \mathbb{Z}_N \), then
\[ \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} \beta v_T'(g_{i,j}) = \int_{a}^{b} (w \beta v_T')(x)dx = (v_T)(b) - (v_T)(a) = 0. \]

Therefore,
\[ v_{N,p-1} = \sum_{i=1}^{N} \sum_{j=1}^{p_i} [v_{i,j}] = \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} \beta v_T'(g_{i,j}) - A_{N,p} \beta v_T'(g_{N,p}) = -A_{N,p} \beta v_T'(g_{N,p}). \]

In other words, we also have
\[ [v_{N,p}] = v_{N,p} - v_{N,p-1} = A_{N,p} \beta v_T'(g_{N,p}). \quad (3.5) \]
Consequently,
\[ |\Pi_h v_T|^2_{L^2} = \sum_{i=1}^{N} \sum_{j=1}^{p} h_i^{-1} [v_{i,j}]^2 = \sum_{i=1}^{N} \sum_{j=1}^{p} h_i^{-1} (A_{i,j} \beta v_T'(g_{i,j}))^2. \]

Noticing that \( A_{i,j} \sim h_i \), we get
\[ (3.6) \quad |\Pi_h v_T|_{L^2} \sim |v_T|_G \sim |v_T|_1. \]

Then the second inequality of (3.3) follows. □

We are now ready to present the inf-sup condition and the continuity of \( a(\cdot, \cdot) \).

**Theorem 3.2.** For all \( u \in H^1, v_T' \in V_T' \), there holds
\[ (3.7) \quad a(u, v_T') \leq M \|u\|_G \|v_T'\|_{T'}. \]

Moreover, if the mesh size \( h \) is sufficiently small, then
\[ (3.8) \quad \inf_{v_T' \in U_T} \sup_{w_T' \in V_T'} \frac{a(v_T', w_T')}{\|v_T'\|_G \|w_T'\|_{T'}} \geq c_0, \]

where both \( M, c_0 \) are constants independent of the mesh-size \( h \). Consequently,
\[ (3.9) \quad \|u - u_T\|_G \leq M \inf_{v_T' \in U_T} \|u - v_T\|_G. \]

**Proof.** By (2.17) and the Cauchy-Schwartz inequality, we have
\[ a(u, v_T') \leq |u|_G \left( \sum_{i=1}^{N} \sum_{j=1}^{p} \frac{\beta}{A_{i,j}} [v_{i,j}]^2 \right)^{\frac{1}{2}} + \max(|\gamma|, |c|) \|u\|_1 \left( \sum_{i=1}^{N} \sum_{j=1}^{p} h_i v_{i,j}^2 \right)^{\frac{1}{2}} \]
\[ \leq M \|u\|_G \|v_T'\|_{T'}, \]

where the constant \( M \) only depends on \( \beta, \gamma, c \). Then (3.7) follows.

Recall the definition of the linear mapping \( \Pi_h \), then we have
\[ a(v_T', \Pi_h v_T) = I_1 + I_2, \quad \forall v_T' \in U_T \]

with
\[ I_1 = \sum_{i=1}^{N} \sum_{j=1}^{p} [v_{i,j}] \beta (g_{i,j}) v_T'(g_{i,j}), \quad I_2 = \sum_{i=1}^{N} \sum_{j=1}^{p} v_{i,j} \int_{g_{i,j}}^{g_{i,j+1}} (\gamma v_T'(x) + cv_T(x)) dx. \]

In light of (3.4), we have
\[ I_1 = \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} (\beta v_T')^2(g_{i,j}) \geq \beta_0 |v_T'|_1^2. \]

To estimate \( I_2 \), we let \( V(x) = \int_a^x (\gamma v_T'(s) + cv_T(s)) ds \) and denote by
\[ E_i = \int_{x_{i-1}}^{x_i} w(x) \beta(x) v_T'(x) V(x) dx = \sum_{j=1}^{p} A_{i,j} (\beta v_T')(g_{i,j}) V(g_{i,j}), \]
the error of Gauss quadrature in the interval \( \tau_i, i \in \mathbb{Z}_N \). Then

\[
I_2 = -\sum_{i=1}^{N} \sum_{j=1}^{p} [v_{i,j}](g_{i,j}) = -\int_{a}^{b} w(x) \beta(x) v'_T(x) V(x) dx + \sum_{i=1}^{N} E_i
\]

\[
= \int_{a}^{b} (\gamma v'_T + cv_T) v_T(x) dx + \sum_{i=1}^{N} E_i = \int_{a}^{b} cv''_T(x) dx + \sum_{i=1}^{N} E_i,
\]

where in the second and last steps, we have used the integration by parts and the fact that \( v_T(a) = v_T(b) = 0 \). On the other hand, the error of Gauss quadrature can be represented as (see, e.g., [19], p98, (2.7.12))

\[
E_i = h_i^{2p+1} (pl)^4 \frac{1}{(2p)!} (\beta v'_T V)^{(2p)}(\xi_i),
\]

where \( \xi_i \in \tau_i \). By the Leibnitz formula of derivatives, we have

\[
| (\beta v'_T V)^{(2p)}(\xi_i) | \leq \sum_{k=p+1}^{2p} \binom{2p}{k} | (\gamma v'_T + cv_T)^{(k-1)}(\beta v'_T)^{(2p-k)}(\xi_i) | \leq c_1 \| v_T \|^2_{p,\infty,\tau_i}
\]

with

\[
c_1 = \max\{\beta, \gamma, c\} \sum_{k=p+1}^{2p} \binom{2p}{k}.
\]

Noticing that \( \beta v'_T \in P_p, k \in \mathbb{Z}_p \), the inverse inequality holds and thus

\[
\| \beta v_T \|_{p,\infty,\tau_i} \lesssim h_i^{-(p+1)} | \beta v_T |_{1,\tau_i}, \quad p \geq 1.
\]

Then

\[
|E_i| \leq c_1 (pl)^4 \frac{1}{(2p)!} h_i^2 |\beta v_T|_{1,\tau_i}^2.
\]

Plugging the estimate for \( E_i \) into the formula of \( I_2 \) yields

\[
I_2 \geq c \| v_T \|^2 - \frac{c_1 (pl)^4}{(2p+1)(2p)!} h^2 |v_T|_{1}^2.
\]

Then for sufficiently small \( h \), we have

\[
\tag{3.10}
a(v_T, \Pi_h v_T) \geq \frac{\beta_0}{2} |v_T|_{1}^2 + \frac{c}{2} \| v_T \|^2 \geq \frac{1}{2} \min\{\beta_0, c\} \| v_T \|^2_{1}.
\]

In light of (3.3)-(3.6), there holds for any \( v_T \in U_T \),

\[
\sup_{w_T \in V_T} \frac{a(v_T, w_T)}{\| w_T \|_{T'}} \geq \frac{a(v_T, \Pi_h v_T)}{\| \Pi_h v_T \|_{T'}} \geq c_0 \| v_T \|_G,
\]

where \( c_0 \) is a constant independent of the mesh size \( h \). The inf-sup condition (3.8) then follows. Combining the continuity (3.7), inf-sup condition (3.8), and the orthogonality of IFVM, we derive (3.9) following similar arguments as in [5] or [41].

**Remark 3.1.** As we may observe in the proof of the above theorem, (3.8) always holds no matter where the interface is. In other words, the inf-sup condition of the IFVM is independent of the location of the interface point.
A direct consequence of the above theorem is the following error estimate for the IFVM.

**Corollary 3.3.** Let \( T = \{ \tau_i \}_{i=1}^N \) be a partition of \( \Omega \) such that the interface \( \alpha \in \tau_k \). Let \( u_T \in U_T \) be the IFV solution of (2.18), and \( u \in W_\beta^{1,\infty}(\Omega) \) be the exact solution of (2.1) - (2.4). Then

\[
|u - u_T|_1 \lesssim h^p |u|_{p+1,\infty}.
\]

**Proof.** Noticing that \( \| \cdot \|_1 \leq \| \cdot \|_G \), we have from (3.9)

\[
\|u - u_T\|_1 \leq \|u - u_T\|_G \leq \frac{M}{c_0} \inf_{v_T \in U_T} \|u - v_T\|_G \leq \frac{M}{c_0} \|u - u_I\|_G,
\]

where \( u_I \) is some interpolation function of \( u \). Then (3.11) follows from the approximation theory of the immersed finite element space [2]. \( \square \)

**4. Superconvergence analysis.** In this section, we derive some superconvergence properties of IFVM. First we introduce a special Guass-Lobatto projection, which is of great importance in the superconvergence analysis. For any \( u \in W_\beta^{m,q}(\Omega), m \geq 1 \), we have the following (generalized) Lobatto expansion of \( u \) on each element \( \tau_i \) [10]:

\[
(4.1) \quad u(x)|_{\tau_i} = \sum_{n=0}^{\infty} u_{i,n} \phi_{i,n}(x),
\]

where

\[
u_{i,0} = u(x_{i-1}), \quad u_{i,1} = u(x_i), \quad u_{i,n} = \int_{\tau_i} \beta u'(x) \phi_{i,n}'(x) dx = \int_{\tau_i} \beta \phi_{i,n}'(x) \phi_{i,n}'(x) dx.
\]

We define the Gauss-Lobatto projection \( I_h : W_\beta^{m,q}(\Omega) \to U_T \) as follows

\[
(4.2) \quad (I_h u)|_{\tau_i} = \sum_{n=0}^{p} u_{i,n} \phi_{i,n}(x).
\]

Let \( \tilde{U}_T = \{ v \in C(\Omega) : v|_{\tau_i} \in \text{span}\{ \phi_{i,n} : n = 0, 1, \ldots, p \}, v(a) = 0 \} \). Then we define a special function \( \omega_T \in \tilde{U}_T \) as follows

\[
(4.3) \quad \beta \omega_T'(g_{i,j}) = \beta(u - I_h u)'(g_{i,j}) - \gamma(u - I_h u)(g_{i,j}), \quad (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_p.
\]

**Lemma 4.1.** Let \( u \in \tilde{W}_\beta^{2p+1,\infty}(\Omega) \) and \( \omega_T \in \tilde{U}_T \) be the special function defined by (4.3). Then \( \omega_T \) is well-defined, and for all \( p \geq 2 \)

\[
(4.4) \quad \| \omega_T \|_{0,\infty} \leq C h^{p+2} \| u \|_{2p+1,\infty},
\]

where \( C \) is a positive constant dependent on the coefficients \( \beta \) and \( \gamma \).

**Proof.** First, \( \beta \omega_T \in \mathbb{P}_{p-1} \) is uniquely determined by the first condition of (4.3) and thus \( \omega_T' \) is well-defined. Since \( \omega_T \) is continuous satisfying \( \omega_T(a) = 0 \), then \( \omega_T \) is uniquely determined. By the approximation property of \( I_h \) (see, [10]), we get

\[
\| u - I_h u \|_{0,\infty} \lesssim h^{p+1} |u|_{p+1,\infty}, \quad \beta(u - I_h u)'(g_{i,j}) \lesssim h^{p+1} |u|_{p+2,\infty},
\]
which gives
\[ \| \beta \omega_T' \|_{0, \infty, \tau_i} \lesssim h^{p+1} \| u \|_{p+2, \infty}. \]

On the other hand, by Gauss quadrature,
\[ \omega_T(x_i) - \omega_T(x_{i-1}) = \int_{\tau_i} \omega_T(x)dx = \sum_{j=1}^{p} A_{i,j} (\beta \omega_T'(g_{i,j})) \]
\[ = \sum_{j=1}^{p} A_{i,j} (\beta(u - I_h u)' + \gamma(u - I_h u)) (g_{i,j}) \]
\[ = \int_{\tau_i} \frac{1}{\beta} (\beta(u - I_h u)' + \gamma(u - I_h u)) (x)dx - E_i, \]
where
\[ E_i = \int_{\tau_i} \frac{1}{\beta} (\beta(u - I_h u)' + \gamma(u - I_h u)) (x)dx - \sum_{j=1}^{p} A_{i,j} (\beta(u - I_h u)' + \gamma(u - I_h u)) (g_{i,j}) \]
denotes the error of Gauss quadrature in \( \tau_i \). By the orthogonality of the Lobatto polynomials, we have \((u - I_h u) \perp P_0(\tau_i), i \neq k\), then
\[ \omega_T(x_i) - \omega_T(x_{i-1}) = \begin{cases} -E_i, & \text{if } i \neq k, \\ \int_{\tau_i} \frac{1}{\beta} (u - I_h u)(x)dx - E_k, & \text{if } i = k. \end{cases} \]
Noticing that
\[ E_i = \int_{\tau_i} \frac{h^{2p+1}(p)!}{(2p+1)(2p)!} (\beta(u - I_h u)' + \gamma(u - I_h u))^{(2p)}(\xi_i), \quad \xi_i \in \tau_i, \]
we have
\[ |E_i| \lesssim h^{2p+1} \| u \|_{2p+1, \infty, \tau_i}, \]
which yields
\[ |\omega_T(x_i) - \omega_T(x_{i-1})| \lesssim h^{2p+1} \| u \|_{2p+1, \infty}, \quad i \neq k, \]
\[ |\omega_T(x_k) - \omega_T(x_{k-1})| \lesssim h^{p+2} \| u \|_{2p+1, \infty}. \]
Using the fact \( \omega_T(a) = \omega_T(b) = 0 \), we have for all \( i \in \mathbb{Z}_N \)
\[ |\omega_T(x_i)| \lesssim h^{p+2} \| u \|_{2p+1, \infty}, \quad p \geq 2. \]

Then for all \( x \in \tau_i \),
\[ |\omega_T(x)| = |\omega_T(x_{i-1}) + \int_{x_{i-1}}^{x} \omega_T'(x)dx| \lesssim h^{p+2} \| u \|_{2p+1, \infty}. \]
This finishes our proof. □

We define a linear interpolant of \( \omega_T \) on \([a, b]\) as follows.

\[ \omega_I(x) = \omega_T(b)C_h \int_a^x \frac{1}{\beta(x)} dx, \quad (4.5) \]
where $C_b = (\frac{\alpha - \alpha}{\beta} + \frac{\beta - \alpha}{\beta})^{-1}$. It is easy to check that
\[ \omega_I(a) = 0 = \omega_T(a), \quad \omega_I(b) = \omega_T(b), \quad [\omega_I(a)] = 0, \quad \left[ \beta \omega_T^j(\tilde{a}) \right] = 0, \quad \forall j = 1, 2, \ldots, p. \]

Apparently, $\omega_I \in \bar{U}_T$ and $\omega_T - \omega_I \in U_T$. Moreover, there holds
\[ |\omega_I(x)| + |\beta \omega_I(x)| \leq |C_b \omega_T(b)| \leq \|\omega_T\|_{0, \infty} \lesssim h^{p+2} \|u\|_{2p+1, \infty}, \quad \forall x \in \Omega. \]

Now we are ready to show our superconvergence properties of the IFVM.

**Theorem 4.2.** Let $\mathcal{T} = \{\tau_i\}_{i=1}^N$ be an partition of $\Omega$ such that the interface $\alpha \in \tau_k$. Let $u_T \in U_T$ be the IFV solution of (2.18) with $p \geq 2$, and $u \in W^{2p+1, \infty}_\beta(\Omega)$ be the exact solution of (2.1) - (2.4). Then

- The IFV solution $u_T$ is superclose to the Gauss-Lobatto projection of the exact solution, i.e.,
  \[ \|u_T - I_h u\|_{0, \infty} = O(h^{p+2}) \]

- The function value approximation of $u_T$ is superconvergent at roots of $\phi_{i,p+1}$ when $p + 2$, with an order of $p + 2$. That is,
  \[ (u - u_T)(l_{i,j}) = O(h^{p+2}), \]
  where $l_{i,j}$ are zeros of $\phi_{i,p+1}$.

- The flux approximation of $\beta u_T$ is superconvergent with an order of $p + 1$ at the Gauss points $g_{i,j}, (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_p$, i.e.,
  \[ \beta (u - u_T)'(g_{i,j}) = O(h^{p+1}). \]

- For diffusion only equation, i.e., $\gamma = c = 0$, there hold
  \[ \beta (u - u_T)'(g_{i,j}) = O(h^{2p}), \quad (u - u_T)(x_i) = O(h^{2p}), \]
  \[ (u - u_T)(x_i) - (u - u_T)(x_{i-1}) = O(h^{2p+1}). \]

**Proof.** First, let
\[ u_I = I_h u + \omega_T - \omega_I, \]
where $\omega_T$ is defined by (4.3), and $\omega_I$ is the linear interpolant of $\omega_T$ given by (4.5), and define a operator $D_x^{-1}$ on all $v \in H^1(\Omega)$,
\[ D_x^{-1}v(x) = \int_a^x v(x)dx. \]

For all $v_T \in V_T$, it follows from (2.17)
\[ a(u - u_I, v_T) = \sum_{i=1}^N \sum_{j=1}^p [v_{i,j}](\beta(u - u_I)' - \gamma(u - u_I) - cD_x^{-1}(u - u_I))(g_{i,j}) \]
\[ = \sum_{i=1}^N \sum_{j=1}^p [v_{i,j}] (\beta \omega_I' + \gamma(\omega_T - \omega_I) - cD_x^{-1}(u - u_I))(g_{i,j}), \]
where in the last step, we have used the definition of $\omega_T$ in (4.3), which yields
\[ (\beta(u - u_I)' - \gamma(u - u_I))(g_{i,j}) = \gamma(\omega_T - \omega_I)(g_{i,j}) + \beta \omega_I'(g_{i,j}). \]
Noticing that \((u - \mathcal{I}_h u) \perp \mathbb{P}_0(\tau), i \neq k\), we have
\[
D_x^{-1}(u-u_I)(x) = \begin{cases} 
\int_{x_{k-1}}^{x} (u - \mathcal{I}_h u)(x) dx - \int_{a}^{x} (\omega_T - \omega_I)(x) dx, \ i \leq k, \\
\int_{x_{k-1}}^{x} (u - \mathcal{I}_h u)(x) dx - \int_{a}^{x} (\omega_T - \omega_I)(x) dx, \ i > k,
\end{cases}
\]
which yields, together with (4.4) and (4.6)
\[
\|D_x^{-1}(u-u_I)\|_{0,\infty} \lesssim h\|u - \mathcal{I}_h u\|_{0,\infty} + \|\omega_T\|_{0,\infty} \lesssim h^{p+2}\|u\|_{2p+1,\infty}.
\]
Then by the Cauchy-Schwartz inequality, (4.4) and (4.6)
\[
|a(u-u_I, v_T)| \lesssim |v_T|_{1,T} \left( \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} (\beta\omega_T' + \gamma(\omega_T - \omega_I) - cD_x^{-1}(u-u_I))^{2}(g_{i,j}) \right)^{\frac{1}{2}}
\]
\[
\lesssim |v_T|_{1,T} \left( \|\beta\omega_T'\|_{0,\infty} + \|\omega_T - \omega_I\|_{0,\infty} + \|D_x^{-1}(u-u_I)\|_{0,\infty} \right)
\lesssim h^{p+2}\|u\|_{2p+1,\infty}\|v_T\|_T, \ \forall v_T' \in V_T.
\]
Now we choose \(v_T = u_I - u_T \in U_T\) in (3.8) and use the orthogonality to obtain
\[
\|u_h - u_I\|_1 \leq \|u_h - u_I\|_G \leq \frac{1}{c_0} \sup_{v_T' \in V_T'} \frac{a(u_h - u_I, v_T')}{\|v_T\|_T} \lesssim h^{p+2}\|u\|_{2p+1,\infty}.
\]
Noticing that \((u_h - u_I)(a) = 0\), we have
\[
(u_h - u_I)(x) = \int_{a}^{x} (u_h - u_I)'(x) dx,
\]
which yields
\[
\|u_h - u_I\|_{0,\infty} \lesssim |u_h - u_I|_1 \lesssim h^{p+2}\|u\|_{2p+1,\infty},
\]
and thus,
\[
\|u_h - \mathcal{I}_h u\|_{\infty} \leq \|u_h - u_I\|_{0,\infty} + \|\omega_T - \omega_I\|_{0,\infty} \lesssim h^{p+2}\|u\|_{2p+1,\infty}.
\]
This finishes the proof of (4.7). Since \(\beta(u_T - \mathcal{I}_h u)' \in \mathbb{P}_{p-1}\), the inverse inequality holds. Then
\[
\|\beta(u_T - \mathcal{I}_h u)\|_{0,\infty} \lesssim h^{-1}\|\beta(u_T - \mathcal{I}_h u)\|_{0,\infty} \lesssim h^{p+1}\|u\|_{2p+1,\infty}.
\]
It has been proved in [10] that
\[
(u - \mathcal{I}_h u)(l_{i,j}) \lesssim h^{p+2}\|u\|_{p+2,\infty}, \ \beta(u - \mathcal{I}_h u)'(g_{i,j}) \lesssim h^{p+1}\|u\|_{p+2,\infty}.
\]
Then (4.8)–(4.9) follow from the triangle inequality.

Now we consider the special case \(\gamma = c = 0\). For simplicity, we denote \(e_u = u - u_T\).
It follows from the FV scheme (2.16) that
\[
\beta e_u'(g_{i,j}) - \beta e_u'(g_{i,j+1}) = 0.
\]
In other words,
\[
\beta e_u'(g_{i,j+1}) = C_0,
\]
where \( C_0 \) is a constant. Summing up all \((i,j)\) yields

\[
C_0 \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} = \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} \beta_e'(g_{i,j}) = \int_a^b e_u'(x)dx - \sum_{i=1}^{N} E_i = -\sum_{i=1}^{N} E_i,
\]

where the error of Gauss quadrature \( E_i \) in each element \( \tau_i \) can be represented as

\[
|E_i| = \frac{h_i^{2p+1}}{(2p+1)(2p)!} \|e_u^{(2p+1)}(\xi_i)\| \lesssim h^{2p+1}\|u\|_{2p+1,\infty},
\]

where \( \xi_i \in \tau_i \) is some point. Noticing that \( \sum_{i=1}^{N} \sum_{j=1}^{p} A_{i,j} \sim (b-a) \), we have

\[
|C_0| \lesssim \frac{1}{b-a} \sum_{i=1}^{N} |E_i| \lesssim h^{2p}\|u\|_{2p+1,\infty},
\]

and thus

\[
|\beta_e'(g_{i,j+1})| = |C_0| \lesssim h^{2p}\|u\|_{2p+1,\infty}.
\]

Again, we use Gauss quadrature to obtain

\[
e_u(x_i) - e_u(x_i-1) = \int_{\tau_i} e_u'(x)dx = \sum_{j=1}^{p} A_{i,j} \beta_e'(g_{i,j}) + E_i = h_i C_0 + E_i,
\]

and thus

\[
e_u(x_j) = e_u(x_0) + C_0 \sum_{i=1}^{j} h_i + \sum_{i=1}^{j} E_i = C_0 \sum_{i=1}^{j} h_i + \sum_{i=1}^{j} E_i.
\]

Combining the estimates for \( C_0 \) and \( E_i \), the desired results (4.10)-(4.11) follow. The proof is complete.

**Remark 4.1.** As a direct consequence of (4.7), we immediately obtain the optimal convergence rate of the IFV solution under the \( L^2 \) norm. That is

\[
\|u - u_T\|_0 \leq \|u - I_h u\|_0 + \|I_h u - u_T\|_0 = O(h^{p+1}).
\]

**Remark 4.2.** The error estimate (3.11) and the superconvergence results (4.7)-(4.11) can be readily extended to interface problems with multiple discontinuity.

**Remark 4.3.** In general, there is no superconvergence behavior on the interface point \( \alpha \), unless it coincides with the generalized Gauss or Lobatto points.

**5. Numerical Examples.** In this section, we present some numerical experiments to demonstrate the features of IFVM.

We test the same example as in [10]. The exact solution is chosen as

\[
u(x) = \begin{cases} 
\frac{1}{\beta^-} \cos(x), & \text{if } x \in [0, \alpha), \\
\frac{1}{\beta^+} \cos(x) + \left(1 - \frac{1}{\beta^+} \right) \cos(\alpha), & \text{if } x \in (\alpha, 1],
\end{cases}
\]
where $\alpha = \pi/6$ is the interface point, and $(\beta^-, \beta^+) = (1, 5)$ represents a moderate discontinuity of the diffusion coefficient.

We use a family of uniform meshes $\{T_h\}$, $h > 0$ where $h$ denotes the mesh size. We test the IFVM for polynomial degrees $p = 1, 2, 3$. Due to the finite machine precision, we choose different sets of meshes for different polynomial degrees $p$. The convergence rate is calculated using linear regression of the errors. Error $e_T = u_T - u$ in the following norms will be calculated.

$$
\|e_T\|_N = \max_{x \in T} |u_T(x) - u(x)|,
\|e_T\|_{0,\infty} = \max_{x \in \Omega} |u_T(x) - u(x)|,
\|e_T\|_L = \max_{x \in \{i_p\}} |u_T(x) - u(x)|,
\|\beta e_T\|_G = \max_{x \in \{ip\}} |\beta u_T(x) - \beta u(x)|,
\|e_T\|_0 = \left( \int_{\Omega} |u_T - u|^2 \, dx \right)^{1/2},
\|e_T\|_1 = \left( \int_{\Omega} |u_T' - u'|^2 \, dx \right)^{1/2},
\|e_T\|_P = \max_{i} |e_T(x_i) - e_T(x_{i-1})|.
$$

Here, $\|e_T\|_N$ denotes the maximum error over all the nodes (mesh points). $\|e_T\|_{0,\infty}$ is the infinity norm over the whole domain $\Omega$. This is computed by choosing 10 uniformly distributed points on each non-interface element, and 10 uniformly distributed points in each sub-element of an interface element, and then calculating the largest discrepancy. $\|\beta e_T\|_G$ is the maximum error of flux over all (generalized) Gauss points. $\|e_T\|_L$ is maximum solution error over all (generalized) Lobatto points. $\|e_T\|_0$ and $\|e_T\|_1$ are the standard Sobolev $L^2$- and semi-$H^1$- norms. $\|e_T\|_P$ measures the maximum of the difference of errors at two consecutive nodes.

Example 5.1. In this example, we test IFVM for the diffusion interface problem, i.e., $\gamma = c = 0$. Errors and convergence rates for linear, quadratic and cubic IFVM solutions are listed in Tables 5.1, 5.2, and 5.3, respectively. The convergence rates are consistent with our theoretical analysis in Theorem 4.2. In particular, we note that for quadratic and cubic IFVM solution, the flux error at Gauss points are of order $O(h^{p+1})$, which is higher than IFEM solution $O(h^{p+1})$ [10].

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|e_T|_N$</th>
<th>$|e_T|_{0,\infty}$</th>
<th>$|\beta e_T|_G$</th>
<th>$|e_T|_0$</th>
<th>$|e_T|_1$</th>
<th>$|e_T|_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3.41e-05</td>
<td>1.92e-03</td>
<td>2.11e-04</td>
<td>9.71e-04</td>
<td>2.51e-02</td>
<td>2.14e-05</td>
</tr>
<tr>
<td>16</td>
<td>8.19e-06</td>
<td>4.81e-04</td>
<td>5.14e-05</td>
<td>2.42e-04</td>
<td>1.25e-02</td>
<td>2.89e-06</td>
</tr>
<tr>
<td>32</td>
<td>2.05e-06</td>
<td>1.20e-04</td>
<td>1.29e-05</td>
<td>6.06e-05</td>
<td>6.26e-03</td>
<td>3.82e-07</td>
</tr>
<tr>
<td>64</td>
<td>5.22e-07</td>
<td>3.01e-05</td>
<td>3.25e-06</td>
<td>1.52e-05</td>
<td>3.14e-03</td>
<td>4.95e-08</td>
</tr>
<tr>
<td>128</td>
<td>1.33e-07</td>
<td>7.53e-06</td>
<td>8.19e-07</td>
<td>3.82e-06</td>
<td>1.58e-03</td>
<td>6.31e-09</td>
</tr>
<tr>
<td>256</td>
<td>3.32e-08</td>
<td>1.88e-06</td>
<td>2.05e-07</td>
<td>9.56e-07</td>
<td>7.88e-04</td>
<td>7.95e-10</td>
</tr>
<tr>
<td>512</td>
<td>8.30e-09</td>
<td>4.71e-07</td>
<td>5.12e-08</td>
<td>2.40e-07</td>
<td>3.94e-04</td>
<td>9.96e-11</td>
</tr>
</tbody>
</table>

Rate | 1.99 | 1.99 | 2.00 | 2.00 | 1.00 | 2.95 |

Example 5.2. In the example, we test the superconvergence behavior for general second-order equation, e.g., $\gamma = 1$ and $c = 1$. Tables 5.4 - 5.6 report the errors and convergence rates of $P_1$, $P_2$, and $P_3$ IFVM approximation, respectively. Again, these data indicate the validity of our theoretical analysis. In Figure 5.1 - 5.3, we plot the solution error and the flux error in a uniform mesh consists of eight elements. Note that the interface $\alpha = \pi/6$, depicted by an black dot, is in the fifth element. The (generalized) Lobatto points and the (generalized) Gauss points are show in red color.
Clearly, we can see that solution errors and flux errors at these special points are much closer to 0, than the majority of the points. This again shows the superconvergence behavior of IFVM.

### Table 5.2
Error of $P_2$ IFVM Solution with $\beta = [1, 5]$, $\alpha = \pi/6$, $\gamma = c = 0$.

<table>
<thead>
<tr>
<th>1/h</th>
<th>$|e_T|_N$</th>
<th>$|e_T|_{0, \infty}$</th>
<th>$|\beta e_T|_L$</th>
<th>$|\beta e_T|_G$</th>
<th>$|e_T|_0$</th>
<th>$|e_T|_1$</th>
<th>$|e_T|_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.80e-09</td>
<td>6.87e-06</td>
<td>2.10e-07</td>
<td>1.79e-08</td>
<td>2.51e-06</td>
<td>1.32e-04</td>
<td>1.80e-09</td>
</tr>
<tr>
<td>16</td>
<td>1.80e-10</td>
<td>8.98e-07</td>
<td>1.32e-08</td>
<td>1.12e-09</td>
<td>3.18e-07</td>
<td>3.33e-05</td>
<td>6.32e-11</td>
</tr>
<tr>
<td>24</td>
<td>3.35e-11</td>
<td>2.70e-07</td>
<td>2.61e-09</td>
<td>2.22e-10</td>
<td>9.46e-08</td>
<td>1.48e-05</td>
<td>8.63e-12</td>
</tr>
<tr>
<td>32</td>
<td>1.11e-11</td>
<td>1.15e-07</td>
<td>8.27e-10</td>
<td>6.97e-11</td>
<td>3.97e-08</td>
<td>8.25e-06</td>
<td>2.07e-12</td>
</tr>
<tr>
<td>40</td>
<td>4.62e-12</td>
<td>5.90e-08</td>
<td>3.39e-10</td>
<td>2.93e-11</td>
<td>2.07e-08</td>
<td>5.38e-06</td>
<td>6.90e-13</td>
</tr>
<tr>
<td>48</td>
<td>2.26e-12</td>
<td>3.55e-08</td>
<td>1.63e-10</td>
<td>1.48e-11</td>
<td>1.21e-08</td>
<td>3.76e-06</td>
<td>2.82e-13</td>
</tr>
<tr>
<td>56</td>
<td>1.27e-12</td>
<td>2.23e-08</td>
<td>8.82e-11</td>
<td>7.94e-12</td>
<td>7.57e-09</td>
<td>2.76e-06</td>
<td>1.35e-13</td>
</tr>
<tr>
<td>rate</td>
<td>3.97</td>
<td>2.95</td>
<td>4.00</td>
<td>3.97</td>
<td>2.98</td>
<td>1.99</td>
<td>4.89</td>
</tr>
</tbody>
</table>

### Table 5.3
Error of $P_3$ IFVM Solution with $\beta = [1, 5]$, $\alpha = \pi/6$, $\gamma = c = 0$.

<table>
<thead>
<tr>
<th>1/h</th>
<th>$|e_T|_N$</th>
<th>$|e_T|_{0, \infty}$</th>
<th>$|\beta e_T|_L$</th>
<th>$|\beta e_T|_G$</th>
<th>$|e_T|_0$</th>
<th>$|e_T|_1$</th>
<th>$|e_T|_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6.00e-12</td>
<td>1.87e-06</td>
<td>7.29e-09</td>
<td>3.91e-11</td>
<td>8.96e-07</td>
<td>3.41e-05</td>
<td>6.00e-12</td>
</tr>
<tr>
<td>5</td>
<td>1.30e-12</td>
<td>7.68e-07</td>
<td>1.93e-09</td>
<td>9.53e-12</td>
<td>3.53e-07</td>
<td>1.69e-05</td>
<td>4.19e-12</td>
</tr>
<tr>
<td>6</td>
<td>5.45e-13</td>
<td>3.71e-07</td>
<td>1.02e-09</td>
<td>3.51e-12</td>
<td>1.77e-08</td>
<td>1.01e-05</td>
<td>6.03e-13</td>
</tr>
<tr>
<td>7</td>
<td>1.99e-13</td>
<td>2.01e-07</td>
<td>4.09e-10</td>
<td>1.31e-12</td>
<td>9.35e-08</td>
<td>6.23e-06</td>
<td>1.41e-13</td>
</tr>
<tr>
<td>8</td>
<td>9.69e-14</td>
<td>1.18e-07</td>
<td>2.50e-10</td>
<td>6.26e-13</td>
<td>5.60e-08</td>
<td>4.27e-06</td>
<td>4.19e-14</td>
</tr>
<tr>
<td>9</td>
<td>4.26e-14</td>
<td>7.34e-08</td>
<td>1.24e-10</td>
<td>3.18e-13</td>
<td>3.45e-08</td>
<td>2.95e-06</td>
<td>2.45e-14</td>
</tr>
<tr>
<td>rate</td>
<td>5.97</td>
<td>3.99</td>
<td>4.88</td>
<td>5.92</td>
<td>4.00</td>
<td>3.00</td>
<td>6.70</td>
</tr>
</tbody>
</table>

### Table 5.4
Error of $P_1$ IFVM Solution with $\beta = [1, 5]$, $\alpha = \pi/6$, $\gamma = 1$, $c = 1$.

<table>
<thead>
<tr>
<th>1/h</th>
<th>$|e_T|_N$</th>
<th>$|e_T|_{0, \infty}$</th>
<th>$|\beta e_T|_L$</th>
<th>$|\beta e_T|_G$</th>
<th>$|e_T|_0$</th>
<th>$|e_T|_1$</th>
<th>$|e_T|_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>7.64e-05</td>
<td>1.92e-03</td>
<td>1.21e-03</td>
<td>9.98e-04</td>
<td>2.51e-02</td>
<td>5.49e-05</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>2.03e-05</td>
<td>4.81e-04</td>
<td>3.05e-04</td>
<td>2.49e-04</td>
<td>1.25e-02</td>
<td>7.76e-06</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>4.56e-06</td>
<td>1.20e-04</td>
<td>7.75e-05</td>
<td>6.22e-05</td>
<td>6.26e-03</td>
<td>9.70e-07</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1.17e-06</td>
<td>3.91e-05</td>
<td>1.95e-05</td>
<td>1.56e-05</td>
<td>3.14e-03</td>
<td>1.25e-07</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>2.81e-07</td>
<td>7.53e-06</td>
<td>4.91e-06</td>
<td>3.91e-06</td>
<td>1.38e-03</td>
<td>1.55e-08</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>7.02e-08</td>
<td>1.88e-06</td>
<td>1.23e-06</td>
<td>9.78e-07</td>
<td>7.88e-04</td>
<td>1.95e-09</td>
<td></td>
</tr>
<tr>
<td>512</td>
<td>1.76e-08</td>
<td>4.71e-07</td>
<td>3.07e-07</td>
<td>2.44e-07</td>
<td>3.94e-04</td>
<td>2.45e-10</td>
<td></td>
</tr>
<tr>
<td>rate</td>
<td>2.02</td>
<td>1.99</td>
<td>1.99</td>
<td>2.00</td>
<td>1.00</td>
<td>2.97</td>
<td></td>
</tr>
</tbody>
</table>

REFERENCES


### Table 5.5
Error of $P_2$ IFVM Solution with $\beta = [1, 5]$, $\alpha = \pi/6$, $\gamma = 1$, $c = 1$.

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|e_T|_N$</th>
<th>$|e_T|_{\infty}$</th>
<th>$|e_T|_L$</th>
<th>$|\beta e_T|_C$</th>
<th>$|e_T|_0$</th>
<th>$|e_T|_1$</th>
<th>$|e_T|_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>5.46e-08</td>
<td>6.68e-06</td>
<td>1.71e-07</td>
<td>6.57e-06</td>
<td>2.51e-06</td>
<td>1.32e-04</td>
<td>2.61e-08</td>
</tr>
<tr>
<td>16</td>
<td>8.84e-09</td>
<td>8.90e-07</td>
<td>1.23e-08</td>
<td>8.95e-06</td>
<td>3.18e-07</td>
<td>3.33e-05</td>
<td>1.39e-09</td>
</tr>
<tr>
<td>24</td>
<td>1.84e-09</td>
<td>2.68e-07</td>
<td>2.49e-09</td>
<td>2.70e-07</td>
<td>9.46e-08</td>
<td>1.48e-05</td>
<td>1.90e-10</td>
</tr>
<tr>
<td>32</td>
<td>2.97e-10</td>
<td>1.14e-07</td>
<td>7.92e-10</td>
<td>1.14e-07</td>
<td>3.97e-08</td>
<td>8.25e-06</td>
<td>3.20e-11</td>
</tr>
<tr>
<td>40</td>
<td>4.62e-11</td>
<td>5.86e-08</td>
<td>3.25e-10</td>
<td>5.90e-08</td>
<td>2.07e-08</td>
<td>5.38e-06</td>
<td>6.69e-12</td>
</tr>
<tr>
<td>48</td>
<td>3.32e-11</td>
<td>3.54e-08</td>
<td>1.58e-10</td>
<td>3.55e-08</td>
<td>1.21e-08</td>
<td>3.76e-06</td>
<td>3.27e-12</td>
</tr>
<tr>
<td>56</td>
<td>4.92e-11</td>
<td>2.22e-08</td>
<td>8.63e-11</td>
<td>2.23e-08</td>
<td>7.57e-09</td>
<td>2.76e-06</td>
<td>2.42e-12</td>
</tr>
<tr>
<td>rate</td>
<td>4.14</td>
<td>2.93</td>
<td>3.91</td>
<td>2.93</td>
<td>2.98</td>
<td>1.99</td>
<td>5.03</td>
</tr>
</tbody>
</table>

### Table 5.6
Error of $P_4$ IFVM Solution with $\beta = [1, 5]$, $\alpha = \pi/6$, $\gamma = 1$, $c = 1$.

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$|e_T|_N$</th>
<th>$|e_T|_{\infty}$</th>
<th>$|e_T|_L$</th>
<th>$|\beta e_T|_C$</th>
<th>$|e_T|_0$</th>
<th>$|e_T|_1$</th>
<th>$|e_T|_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6.56e-09</td>
<td>1.89e-06</td>
<td>9.81e-08</td>
<td>2.02e-06</td>
<td>8.95e-07</td>
<td>3.41e-05</td>
<td>3.55e-09</td>
</tr>
<tr>
<td>6</td>
<td>1.82e-09</td>
<td>3.74e-07</td>
<td>1.29e-08</td>
<td>4.03e-07</td>
<td>1.77e-07</td>
<td>1.01e-05</td>
<td>7.17e-10</td>
</tr>
<tr>
<td>8</td>
<td>6.56e-10</td>
<td>1.18e-07</td>
<td>3.30e-09</td>
<td>1.28e-07</td>
<td>5.60e-08</td>
<td>4.27e-06</td>
<td>2.01e-10</td>
</tr>
<tr>
<td>10</td>
<td>2.56e-10</td>
<td>4.85e-08</td>
<td>1.13e-09</td>
<td>5.24e-08</td>
<td>2.30e-08</td>
<td>2.19e-06</td>
<td>6.42e-11</td>
</tr>
<tr>
<td>12</td>
<td>9.88e-11</td>
<td>2.34e-08</td>
<td>4.52e-10</td>
<td>2.52e-08</td>
<td>1.11e-08</td>
<td>1.27e-06</td>
<td>2.09e-11</td>
</tr>
<tr>
<td>14</td>
<td>3.58e-11</td>
<td>1.26e-08</td>
<td>2.00e-10</td>
<td>1.36e-08</td>
<td>5.98e-09</td>
<td>7.95e-07</td>
<td>6.53e-12</td>
</tr>
<tr>
<td>16</td>
<td>1.20e-11</td>
<td>7.39e-09</td>
<td>7.90e-11</td>
<td>7.96e-09</td>
<td>3.50e-09</td>
<td>5.32e-07</td>
<td>1.93e-12</td>
</tr>
<tr>
<td>18</td>
<td>3.09e-12</td>
<td>4.61e-09</td>
<td>5.09e-11</td>
<td>4.96e-09</td>
<td>2.18e-09</td>
<td>3.73e-07</td>
<td>4.40e-13</td>
</tr>
<tr>
<td>rate</td>
<td>4.88</td>
<td>4.00</td>
<td>4.99</td>
<td>4.00</td>
<td>4.00</td>
<td>3.00</td>
<td>5.77</td>
</tr>
</tbody>
</table>
Fig. 5.1. Error and flux error of $P_1$ IFVM solution. $\beta = \{1, 5\}$, $\alpha = \frac{\pi}{6}$

Fig. 5.2. Error and flux error of $P_2$ IFVM solution. $\beta = \{1, 5\}$, $\alpha = \frac{\pi}{6}$


Fig. 5.3. Error and flux error of $P_3$ IFVM solution. $\beta = \{1, 5\}$, $\alpha = \frac{\pi}{6}$.